Introduction to Geostatistics

4. Probability I: events, independence, conditional probability

Edzer J. Pebesma

edzer.pebesma@uni-muenster.de
Institute for Geoinformatics (**ifgi**)
University of Münster

summer semester 2007/8,
April 28, 2008

ifgi

Probability–what is it?

When there's anything we don't know for sure, we can use probability to describe our knowledge of it.
The usual interpretation of probability is that of (limiting) frequency:

- if something happens with probability 0.3, after many repetitions we expect it to happen 30% of the times.
- this implies (assumes) there is some underlying, generating process, which can provide the repetition mechanism.
- sometimes this process is within our control (random sampling), sometimes it is not (and is "nature" said to behave randomly)

Probability–what is it?

When there's anything we don't know for sure, we can use probability to describe our knowledge of it.
The usual interpretation of probability is that of (limiting) frequency:

- if something happens with probability 0.3, after many repetitions we expect it to happen 30% of the times.
- this implies (assumes) there is some underlying, generating process, which can provide the repetition mechanism.
- sometimes this process is within our control (random sampling), sometimes it is not (and is "nature" said to behave randomly)

ifgi

Probability–what is it?

When there's anything we don't know for sure, we can use probability to describe our knowledge of it.
The usual interpretation of probability is that of (limiting) frequency:

- if something happens with probability 0.3, after many repetitions we expect it to happen 30% of the times.
- this implies (assumes) there is some underlying, generating process, which can provide the repetition mechanism.
- sometimes this process is within our control (random sampling), sometimes it is not (and is "nature" said to behave randomly)

Motivation from philosophy

Ludwig Wittgenstein, *Tractatus logico-philosophicus*:

4.464 A tautology's truth is certain, a proposition's possible, a contradiction's impossible. (Certain, possible, impossible: here we have the first indication of the scale that we need in the theory of probability.)

5.153 A proposition is in itself neither probable nor improbable. An event occurs or does not occur, there is no middle course.

5.156 It is in this way that probability is a generalization. It involves a general description of a propositional form. We use probability only in default of certainty—if our knowledge of a fact is not indeed complete, but we do know something about its form. ...

Motivation from philosophy

Ludwig Wittgenstein, *Tractatus logico-philosophicus*:

4.464 A tautology's truth is certain, a proposition's possible, a contradiction's impossible. (Certain, possible, impossible: here we have the first indication of the scale that we need in the theory of probability.)

5.153 A proposition is in itself neither probable nor improbable. An event occurs or does not occur, there is no middle course.

5.156 It is in this way that probability is a generalization. It involves a general description of a propositional form. We use probability only in default of certainty—if our knowledge of a fact is not indeed complete, but we do know something about its form. ...

Motivation from philosophy

Ludwig Wittgenstein, *Tractatus logico-philosophicus*:

4.464 A tautology's truth is certain, a proposition's possible, a contradiction's impossible. (Certain, possible, impossible: here we have the first indication of the scale that we need in the theory of probability.)

5.153 A proposition is in itself neither probable nor improbable. An event occurs or does not occur, there is no middle course.

5.156 It is in this way that probability is a generalization. It involves a general description of a propositional form. We use probability only in default of certainty—if our knowledge of a fact is not indeed complete, but we do know something about its form. ...

Example

- Suppose we have a urn with balls that can be red or blue.
- Person A claims that the urn contains 50% red, and 50% blue balls
- Without looking, we randomly take $n = 10$ balls out of the urn.
- All of the balls turn out to be red.
- What is the probability that this happens if person A is right?
- Should we believe person A?
- How large should n be (with unicolored or red-only) balls to start disbelieving A?
- Given n, how many balls of the same color do we allow before we reject A's claim?

Example

- Suppose we have a urn with balls that can be red or blue.
- Person A claims that the urn contains 50% red, and 50% blue balls
- Without looking, we randomly take $n = 10$ balls out of the urn.
- All of the balls turn out to be red.
- What is the probability that this happens if person A is right?
- Should we believe person A?
- How large should n be (with unicolored or red-only) balls to start disbelieving A?
- Given n, how many balls of the same color do we allow before we reject A's claim?

Example

- Suppose we have a urn with balls that can be red or blue.
- Person A claims that the urn contains 50% red, and 50% blue balls
- Without looking, we randomly take $n = 10$ balls out of the urn.
- All of the balls turn out to be red.
- What is the probability that this happens if person A is right?
- Should we believe person A?
- How large should n be (with unicolored or red-only) balls to start disbelieving A?
- Given n, how many balls of the same color do we allow before we reject A's claim?

Example

- Suppose we have a urn with balls that can be red or blue.
- Person A claims that the urn contains 50% red, and 50% blue balls
- Without looking, we randomly take $n = 10$ balls out of the urn.
- All of the balls turn out to be red.
- What is the probability that this happens if person A is right?
- Should we believe person A?
- How large should n be (with unicolored or red-only) balls to start disbelieving A?
- Given n, how many balls of the same color do we allow before we reject A's claim?

Example

- Suppose we have a urn with balls that can be red or blue.
- Person A claims that the urn contains 50% red, and 50% blue balls
- Without looking, we randomly take $n = 10$ balls out of the urn.
- All of the balls turn out to be red.
- What is the probability that this happens if person A is right?
- Should we believe person A?
- How large should n be (with unicolored or red-only) balls to start disbelieving A?
- Given n, how many balls of the same color do we allow before we reject A's claim?

Example

- Suppose we have a urn with balls that can be red or blue.
- Person A claims that the urn contains 50% red, and 50% blue balls
- Without looking, we randomly take $n = 10$ balls out of the urn.
- All of the balls turn out to be red.
- What is the probability that this happens if person A is right?
- Should we believe person A?
- How large should n be (with unicolored or red-only) balls to start disbelieving A?
- Given n, how many balls of the same color do we allow before we reject A's claim?

Example

- Suppose we have a urn with balls that can be red or blue.
- Person A claims that the urn contains 50% red, and 50% blue balls
- Without looking, we randomly take $n = 10$ balls out of the urn.
- All of the balls turn out to be red.
- What is the probability that this happens if person A is right?
- Should we believe person A?
- How large should n be (with unicolored or red-only) balls to start disbelieving A?
- Given n, how many balls of the same color do we allow before we reject A's claim?

Example

- Suppose we have a urn with balls that can be red or blue.
- Person A claims that the urn contains 50% red, and 50% blue balls
- Without looking, we randomly take $n = 10$ balls out of the urn.
- All of the balls turn out to be red.
- What is the probability that this happens if person A is right?
- Should we believe person A?
- How large should n be (with unicolored or red-only) balls to start disbelieving A?
- Given n, how many balls of the same color do we allow before we reject A's claim?

Looking in detail

- We have a urn with balls: the population
- we draw one or more balls from it: the sample
- based on the sample, we try to infer properties of the population
- Detail: is every ball taken randomly?
- Detail: is every draw independent?
- Independence: drawing with replacement!

Looking in detail

- We have a urn with balls: the population
- we draw one or more balls from it: the sample
- based on the sample, we try to infer properties of the population
- Detail: is every ball taken randomly?
- Detail: is every draw independent?
- Independence: drawing with replacement!

Looking in detail

- We have a urn with balls: the population
- we draw one or more balls from it: the sample
- based on the sample, we try to infer properties of the population
- Detail: is every ball taken randomly?
- Detail: is every draw independent?
- Independence: drawing with replacement!

Looking in detail

- We have a urn with balls: the population
- we draw one or more balls from it: the sample
- based on the sample, we try to infer properties of the population
- Detail: is every ball taken randomly?
- Detail: is every draw independent?
- Independence: drawing with replacement!

Looking in detail

- We have a urn with balls: the population
- we draw one or more balls from it: the sample
- based on the sample, we try to infer properties of the population
- Detail: is every ball taken randomly?
- Detail: is every draw independent?
- Independence: drawing with replacement!

Looking in detail

- We have a urn with balls: the population
- we draw one or more balls from it: the sample
- based on the sample, we try to infer properties of the population
- Detail: is every ball taken randomly?
- Detail: is every draw independent?
- Independence: drawing with replacement!

The probability measure

Probability as a mathematical construct. Let $\Pr(E)$ [S&H: $P(E)$] be the probability of event E.

- We define an event as something that happens, subject to probability. Say, we draw a ball, at random. If A was right, we can say

$$\Pr(\text{ball is red}) = 0.5$$

- Probability, as a measure, has three properties:
  1. The empty (not realizable) event, $\emptyset$, has probability zero:

$$\Pr(\emptyset) = 0$$

  2. Any event has nonnegative probability:

$$\Pr(E) >= 0$$

  3. Any possible event has positive probability:

$$\Pr(E) > 0$$

The probability measure

Probability as a mathematical construct. Let $\Pr(E)$ [S&H: $P(E)$] be the probability of event E.

- We define an event as something that happens, subject to probability. Say, we draw a ball, at random. If A was right, we can say

$$\Pr(\text{ball is red}) = 0.5$$

- Probability, as a measure, has three properties:
  1. The empty (not realizable) event, $\emptyset$, has probability zero:

$$\Pr(\emptyset) = 0$$

  2. Any event has nonnegative probability:

$$\Pr(E) >= 0$$

  3. Any possible event has positive probability:

$$\Pr(E) > 0$$

The probability measure

Probability as a mathematical construct. Let $\Pr(E)$ [S&H: $P(E)$] be the probability of event E.

- We define an event as something that happens, subject to probability. Say, we draw a ball, at random. If A was right, we can say

$$\Pr(\text{ball is red}) = 0.5$$

- Probability, as a measure, has three properties:
  1. The empty (not realizable) event, $\emptyset$, has probability zero:

$$\Pr(\emptyset) = 0$$

  2. Any event has nonnegative probability:

$$\Pr(E) >= 0$$

  3. Any possible event has positive probability:

$$\Pr(E) > 0$$

The probability measure

Probability as a mathematical construct. Let $\Pr(E)$ [S&H: $P(E)$] be the probability of event E.

- We define an event as something that happens, subject to probability. Say, we draw a ball, at random. If A was right, we can say

$$\Pr(\text{ball is red}) = 0.5$$

- Probability, as a measure, has three properties:
  1. The empty (not realizable) event, $\emptyset$, has probability zero:

  $$\Pr(\emptyset) = 0$$

  2. Any event has nonnegative probability:

  $$\Pr(E) >= 0$$

  3. Any possible event has positive probability:

  $$\Pr(E) > 0$$

The probability measure

Probability as a mathematical construct. Let $\Pr(E)$ [S&H: $P(E)$] be the probability of event E.

- We define an event as something that happens, subject to probability. Say, we draw a ball, at random. If A was right, we can say
$$\Pr(\text{ball is red}) = 0.5$$
- Probability, as a measure, has three properties:
  1. The empty (not realizable) event, $\emptyset$, has probability zero:
  $$\Pr(\emptyset) = 0$$
  2. Any event has nonnegative probability:
  $$\Pr(E) >= 0$$
  3. Any possible event has positive probability:
  $$\Pr(E) > 0$$

The R function runif

runif draws one or more random uniform values between 0 and 1:

```
> runif(1)
[1] 0.844497
> runif(10)
 [1] 0.92522851 0.43127488 0.62122534 0.50114290 0.16592987 0.96754708
 [7] 0.02003596 0.07540448 0.79720108 0.36114214
> 0.5 < 0.25
[1] FALSE
> runif(5) < 0.25
[1] FALSE FALSE FALSE  TRUE FALSE
> runif(5) < 0.25
[1]  TRUE FALSE FALSE FALSE FALSE
```

Probability and frequencies

```
> mean(runif(1) < 0.25)
[1] 0
> mean(runif(10) < 0.25)
[1] 0.2
> mean(runif(100) < 0.25)
[1] 0.29
> mean(runif(1000) < 0.25)
[1] 0.234
> mean(runif(10000) < 0.25)
[1] 0.2469
> mean(runif(1e+05) < 0.25)
[1] 0.25065
```

The probability of more than one event

- "AND": $Pr(E_1 \cap E_2)$
- for mutually exclusive events: $Pr(E_1 \cap E_2) = 0$
- "OR": $Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2) - Pr(E_1 \cap E_2)$
- For mutually exclusive, probabilities sum to unity

$$\sum_{i=1}^{n} \Pr(E_i) = 1$$

Summarizing:

$$0 \leq \Pr(E_i) \leq 1$$

$$\Pr(\cup_{all} E_i) = 1$$

$$\Pr(\emptyset) = 0$$

The probability of more than one event

- "AND": $Pr(E_1 \cap E_2)$
- for mutually exclusive events: $Pr(E_1 \cap E_2) = 0$
- "OR": $Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2) - Pr(E_1 \cap E_2)$
- For mutually exclusive, probabilities sum to unity

$$\sum_{i=1}^{n} \mathrm{Pr}(E_i) = 1$$

Summarizing:

$$0 \leq \mathrm{Pr}(E_i) \leq 1$$

$$\mathrm{Pr}(\cup_{all} E_i) = 1$$

$$\mathrm{Pr}(\emptyset) = 0$$

The probability of more than one event

- "AND": $Pr(E_1 \cap E_2)$
- for mutually exclusive events: $Pr(E_1 \cap E_2) = 0$
- "OR": $Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2) - Pr(E_1 \cap E_2)$
- For mutually exclusive, probabilities sum to unity

$$\sum_{i=1}^{n} \mathrm{Pr}(E_i) = 1$$

Summarizing:

$$0 \leq \mathrm{Pr}(E_i) \leq 1$$

$$\mathrm{Pr}(\cup_{all} E_i) = 1$$

$$\mathrm{Pr}(\emptyset) = 0$$

The probability of more than one event

- "AND": $Pr(E_1 \cap E_2)$
- for mutually exclusive events: $Pr(E_1 \cap E_2) = 0$
- "OR": $Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2) - Pr(E_1 \cap E_2)$
- For mutually exclusive, probabilities sum to unity

$$\sum_{i=1}^{n} \mathrm{Pr}(E_i) = 1$$

Summarizing:

$$0 \leq \mathrm{Pr}(E_i) \leq 1$$

$$\mathrm{Pr}(\cup_{all} E_i) = 1$$

$$\mathrm{Pr}(\emptyset) = 0$$

The probability of more than one event

- "AND": $Pr(E_1 \cap E_2)$
- for mutually exclusive events: $Pr(E_1 \cap E_2) = 0$
- "OR": $Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2) - Pr(E_1 \cap E_2)$
- For mutually exclusive, probabilities sum to unity

$$\sum_{i=1}^{n} \mathrm{Pr}(E_i) = 1$$

Summarizing:

$$0 \leq \mathrm{Pr}(E_i) \leq 1$$

$$\mathrm{Pr}(\cup_{all} E_i) = 1$$

$$\mathrm{Pr}(\emptyset) = 0$$

The probability of more than one event

- "AND": $Pr(E_1 \cap E_2)$
- for mutually exclusive events: $Pr(E_1 \cap E_2) = 0$
- "OR": $Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2) - Pr(E_1 \cap E_2)$
- For mutually exclusive, probabilities sum to unity

$$\sum_{i=1}^{n} \mathrm{Pr}(E_i) = 1$$

Summarizing:

$$0 \leq \mathrm{Pr}(E_i) \leq 1$$

$$\mathrm{Pr}(\cup_{all} E_i) = 1$$

$$\mathrm{Pr}(\emptyset) = 0$$

Conditional probabilities.

The probability of event E_2, given that E_1 happens, is the so-called conditional probability, and is defined as

$$\Pr(E_2|E_1) = \frac{\Pr(E_2 \cap E_1)}{\Pr(E_1)}$$

we can read it as the probability of E_2 happening, conditional to event E_1 happening.

Conditional probabilities.

```
> t1 = table(students[c("I.am.2", "Gender")])
> prop.table(t1)

              Gender
I.am.2          female  male
  small          0.128 0.047
  intermediate   0.163 0.360
  tall           0.093 0.209

> prop.table(t1, 1)

              Gender
I.am.2          female male
  small           0.73 0.27
  intermediate    0.31 0.69
  tall            0.31 0.69

> prop.table(t1, 2)

              Gender
I.am.2          female  male
  small          0.333 0.075
  intermediate   0.424 0.585
  tall           0.242 0.340
```

Independence

We know that

$$\Pr(E_2|E_1) = \frac{\Pr(E_2 \cap E_1)}{\Pr(E_1)}$$

Two events E_1 and E_2 with $\Pr(E_1) > 0$ are said to be *stochastically independent* if

$$\Pr(E_2|E_1) = \Pr(E_2)$$

which implies that

$$\Pr(E_2 \cap E_1) = \Pr(E_1)\Pr(E_2)$$

Independence

```
> ab = rep(c("AA", "BB"), 8)
> cd = rep(c("CC", "DD"), each = 8)
> t1 = table(ab, cd)
> t1

    cd
ab   CC DD
  AA  4  4
  BB  4  4

> prop.table(t1, 1)

    cd
ab    CC  DD
  AA 0.5 0.5
  BB 0.5 0.5

> prop.table(t1, 2)

    cd
ab    CC  DD
  AA 0.5 0.5
  BB 0.5 0.5
```