Introduction to Geostatistics

5. Probability II: random variables and probability distributions

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Random variables

- A random variable is a numerical variable whose outcome is subject to chance.
- We say (loosely) that the probability of taking on a certain value is denoted by its *probability density* $p(x)$ (or $f(x)$).
- Examples of discrete variables: throwing a dice, throwing a coin
- Examples of continuous variables: *exact* body length of a randomly sampled person
- If outcomes were completely predictable, there would be no element of chance
- How can we describe probability distributions?
- Discrete random variables, continuous random variables

Probability density and distribution function

Probability *density* $f(x)$ gives, for discrete variables, the amount of probability of being x, and is non-negative: $f(x) = Pr(X = x)$
Probability *distribution* ranges from 0 to 1, and gives the cumulative probability up to x.
For discrete variables

$$F(x_i) = \sum_{x \leq x_i} f(x)$$

for continuous variables

$$F(x_i) = \int_{-\infty}^{x_i} f(x)dx$$

Expectation, Variance

Expectation is the mean value for a random variable. Discrete RV:

$$\text{E}(X) = \mu = \sum_{i=1}^{n} x_i f(x_i)$$

Continuous RV:

$$\text{E}(X) = \mu = \int_{-\infty}^{+\infty} xf(x)dx$$

note that $\text{E}(X)$ is a numeric value, i.e. is non-random, and that the argument of $\text{E}(\cdot)$ is random.
How does the expectation of X relate to the sample mean, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$?
For *random* sampling a sample of size n from an infinite population, each $f(x_i)$ is estimated by $\frac{1}{n}$, and $\hat{\mu} = \bar{x}$.

Variance, covariance

variance of a random variable is defined in terms of expectation

$$\mathrm{Var}(X) = \mathrm{E}(X - \mathrm{E}(X))^2$$

covariance is a measure of co-variation of two random variables

$$\mathrm{Cov}(X, Y) = \mathrm{E}((X - \mathrm{E}(X))(Y - \mathrm{E}(Y)))$$

with the following properties:

$$|\mathrm{Cov}(X, Y)| \leq \sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}$$

if X and Y are stochastically independent, then $\mathrm{Cov}(X, Y) = 0$ (the reverse does not hold)

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Covariance, correlation

If

$$\mathrm{Cov}(X, Y) = \mathrm{E}((X - \mathrm{E}(X))(Y - \mathrm{E}(Y)))$$

and

$$|\mathrm{Cov}(X, Y)| \leq \sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}$$

then the *correlation coefficient* between X and Y,

$$r(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$

must have the property that

$$-1 \leq r(X, Y) \leq 1$$

it measures strength of *linear* relationship, and is 0 in absence of a linear relation, 1 (-1) if the relationship is perfect, ascending (descending).

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Moments

The k-th moment of X is defined as

$$\mu'_k = \mathrm{E}(X^k)$$

The k-th central moment of X is defined as

$$\mu_k = \mathrm{E}((X - \mathrm{E}(X))^k)$$

One can define a probability density function by all its moments. The third central moment is of interest, as it tells whether a distribution is symmetric ($\mu_3 = 0$), or *skew*. Is it right-skew, then $\mu_3 > 0$, is it left-skew then $\mu_3 < 0$.

Bernoulli distribution

$$X = \begin{cases} 1 & \text{red ball} \\ 0 & \text{blue ball} \end{cases}$$

$$f(k) = \begin{cases} q = 1 - p & \text{for} & k = 0 \\ p & \text{for} & k = 1 \\ 0 & \text{otherwise} & \end{cases}$$

p is the probability of success (1), q the probability of failure.

Binomial distribution

From n independent observations of a Bernouilli process having success probability p, we obtain exactly k hits with probability

$$f(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{(n-k)} & \text{for} k = 0, 1, ..., n \\ 0 & \text{otherwise} \end{cases}$$

example: random drawing n balls from a bowl, with replacement; what is the probability of drawing exactly k red balls.

```
> pbinom(4, 9, 0.5)
[1] 0.5
> dbinom(4, 9, 0.5)
[1] 0.2460938
> rbinom(20, 9, 0.5)
 [1] 6 7 4 5 5 3 7 2 2 6 4 5 5 6 2 4 3 6 4 7
```

Poisson distribution

Special case of the Binomial distribution, where $n \to \infty$, and the *rate* $np = \lambda$ is known. The Poisson distribution describes the expected frequencies of "hits": $Pr(X = x|\lambda) = f(x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$
Examples:

- length of a row in a shop (queueing problem)
- discrete events in temporal processes, spatial processes
- usually the Poisson is the base-line case, against which more structured processes are investigated

Uniform distribution

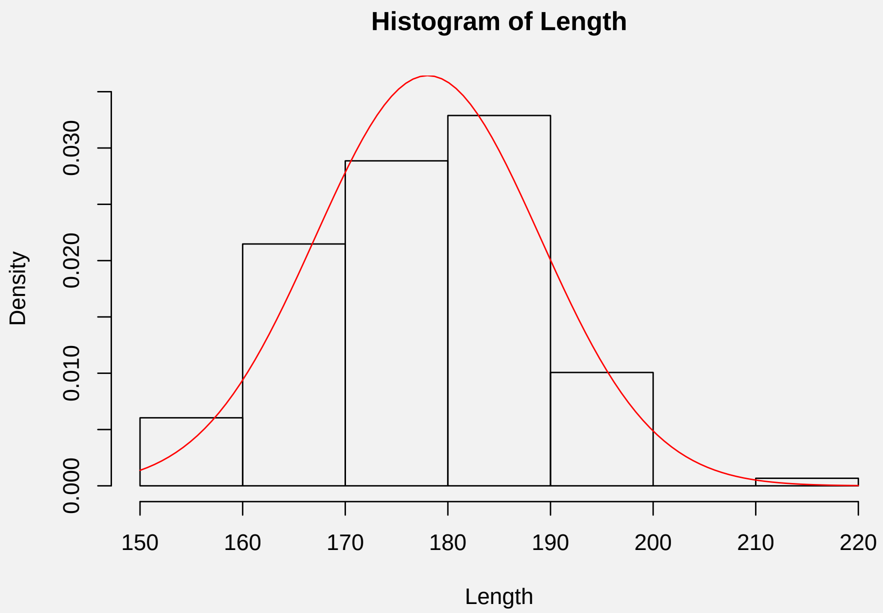
the uniform distribution has a uniform density between its minimum value a and maximum value b:

$$f(x) = \begin{cases} 1/(b-a) & \text{for} \quad a < x < b \\ 0 & \text{for} \quad x < a \text{ or } x > b \end{cases}$$

```
> runif(n = 10, min = 25, max = 50)

 [1] 35.71244 49.27612 41.59993 30.49697 32.38869 47.24659 47.25337
 [8] 26.49742 39.28409 48.85743

> punif(9, min = 0, max = 10)

[1] 0.9
```

Normal (Gaussian) distribution

```
> m = mean(Length)
> s = sqrt(var(Length))
> r = 150:220
> hist(Length, probability = TRUE, ylim = c(0, 0.035))
> curve(dnorm(x, m, s), add = TRUE, col = "red")
```



Gaussian density function

Do not remember this:

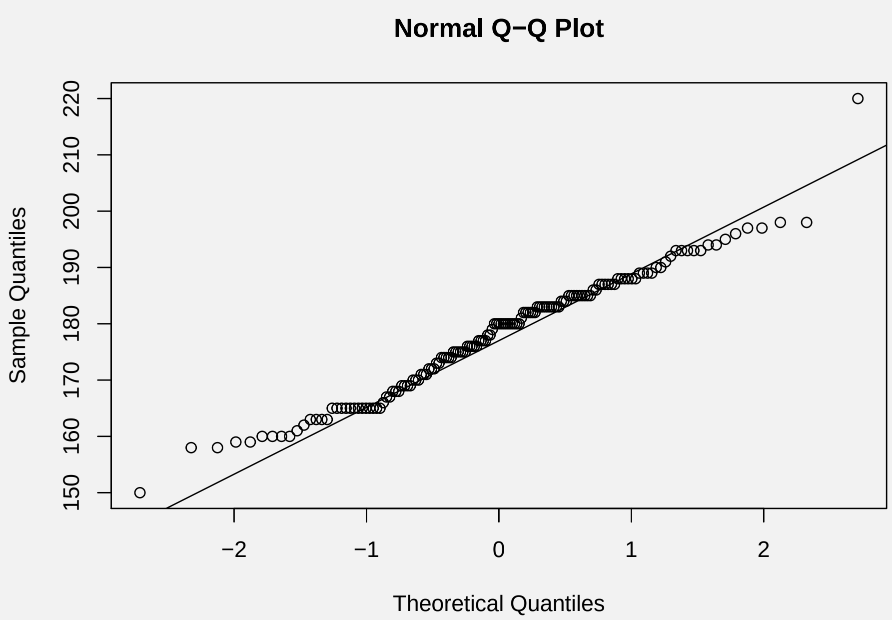
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

But remember:

- *only* depends on μ and σ
- mean μ is also median: symmetric
- ranges from $-\infty$ to ∞
- approx. 68% lies between $\mu - \sigma$ and $\mu + \sigma$
- approx. 95% lies between $\mu - 2\sigma$ and $\mu + 2\sigma$

Normal probability plot

```
> qqnorm(Length)
> qqline(Length)
```



Normal probability plot (2)

```
> qqnorm(Length)
> qqline(Length)
> x = c(-3, 3)
> y = c(m - 3 * s, m + 3 * s)
> lines(cbind(x, y), col = "red")
```