Introduction to Geostatistics

6. Sampling: strategies, point estimation, confidence intervals

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Sampling

Sampling can be

- Random (any population elements can enter the sample with a given, non-zero probability)
- Non-random (population elements enter the sample with an unknown, or with zero probability)

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Examples of non-random sampling

Three examples:

- no random process was used to generate the sampling locations
- part of the area had zero inclusion probability.
  Suppose we want to sample vegetation in NRW. For this, we choose 100 sites randomly, but in forests only. We can now estimate e.g. which fraction of the NRW forests are deciduous (population: forests in NRW), but we cannot estimate which area of NRW consists of forests (population: all of NRW; the non-forest locations had zero inclusion probability in the sample.)

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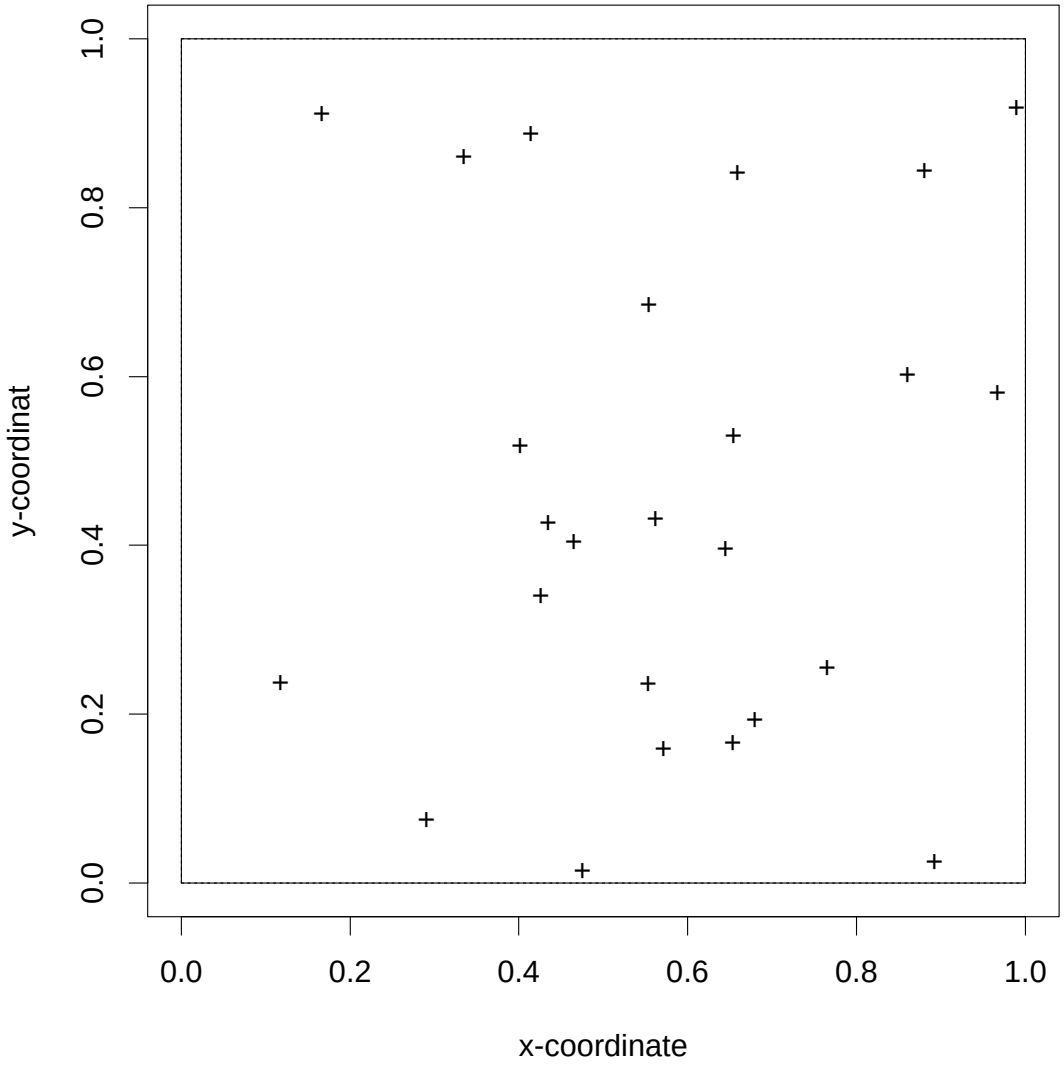
simple random sampling, n = 25 , blocks = 1



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Simple random sampling (1)

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> plot(nrw, col = "lightgrey")
> pts = spsample(nrw, 100, "random")
> points(pts, pch = 3)
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Simple random sampling (2)

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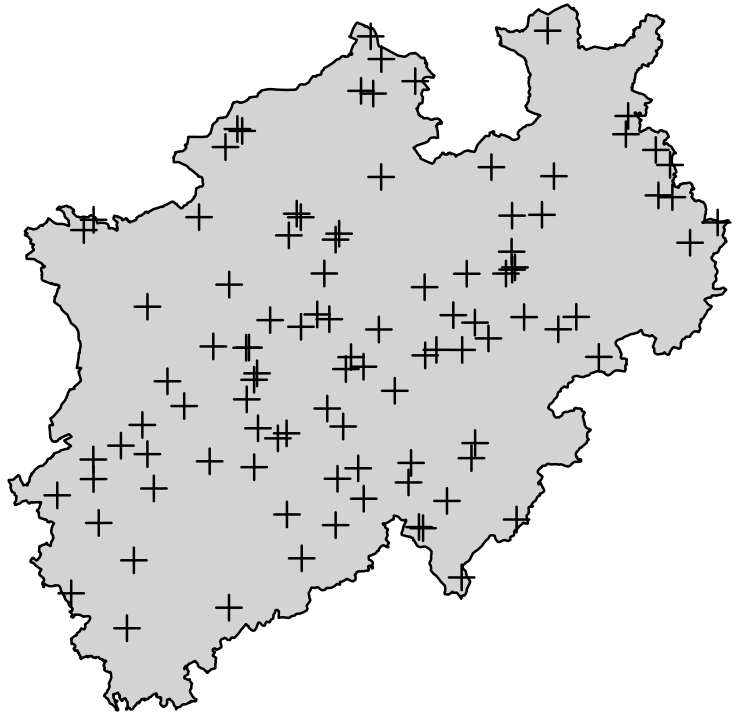

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Simple random sampling (3)

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> plot(nrw, col = "lightgrey")
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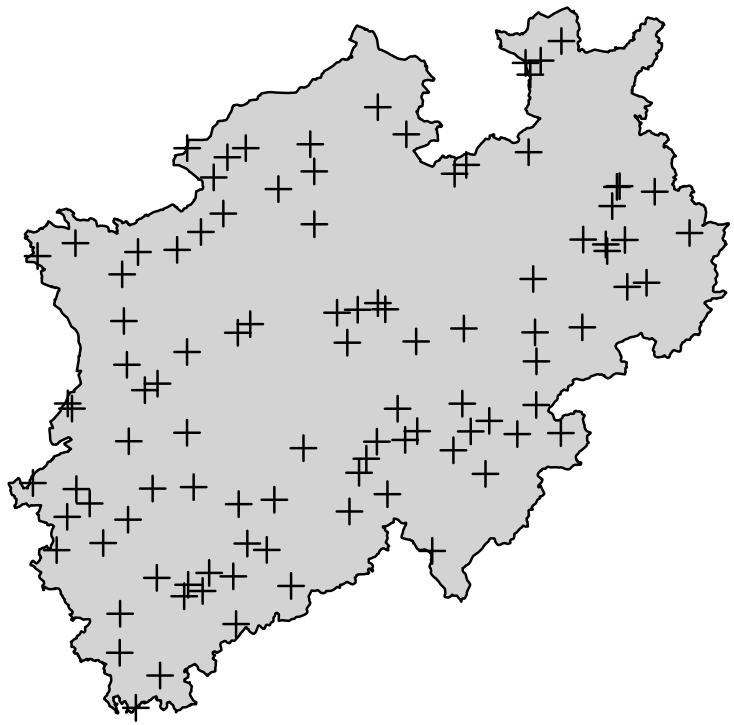

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Simple random sampling (4)

```
> plot(nrw, col = "lightgrey")
> pts = spsample(nrw, 100, "random")
> points(pts, pch = 3)
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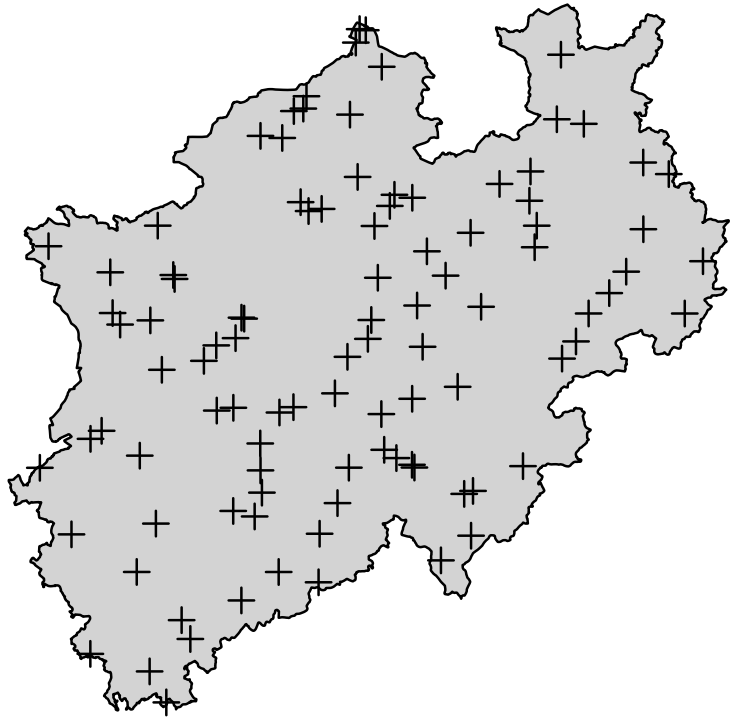

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Simple random sampling (5)

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> plot(nrw, col = "lightgrey")
> pts = spsample(nrw, 100, "random")
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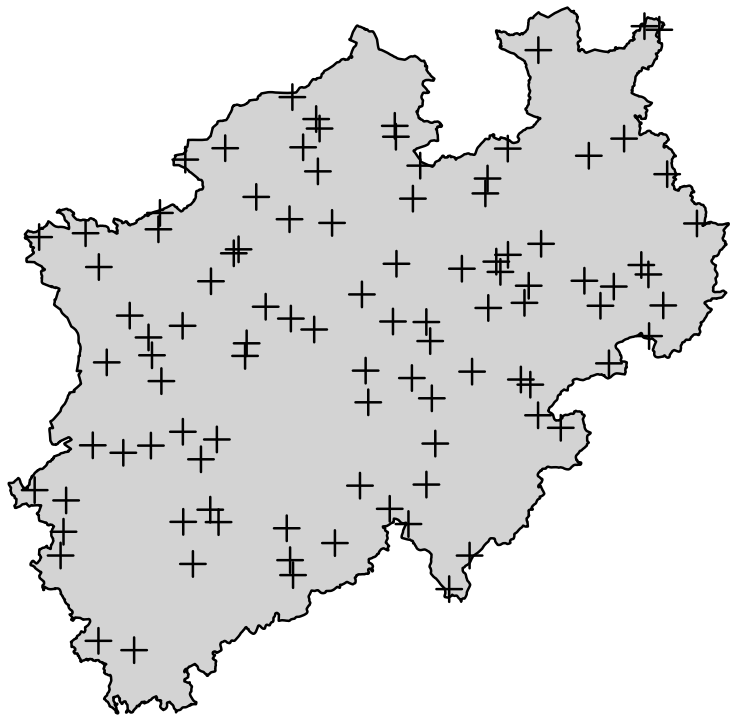

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Properties and strategy

with simple random sampling,

- every element has identical probability to enter the population
- every point is drawn independently from the others
- as the number of points in an area is infinite, replacement is not an issue (in theory; in practice, the choice may be quite constrained)
- this makes the sample elements completely independent (in a *design-based perspective*).
- How to do this? For an area A, consider its bounding box B. For n points, repeat
  - draw a uniform random coordinate x' from the x range of B
  - draw a uniform random coordinate y' from the y range of B
  - accept (x', y') if it is inside (or on) A (point-in-polygon)
  - until we have accepted n points

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Other spatial sampling approaches

- Random sampling over another agent then space: (e.g. if you randomly sample people, the spatial pattern of selected persons will follow the population density pattern.)
- Spatial random sampling that uses map information to vary densities (e.g., for a bird inventory sample forest locations with 0.05 pts/km, agricultural locations with 0.01 pts/km, urban areas with 0.005 pts/km; density may depends on expected variety and viewing conditions)
- Spatially homogeneous, but non-simple random sampling (Ripley, 1981, Spatial Statistics):
  - Stratified random sampling
  - Regular (systematically aligned) sampling
  - Non-aligned regular sampling
  - Clustered sampling

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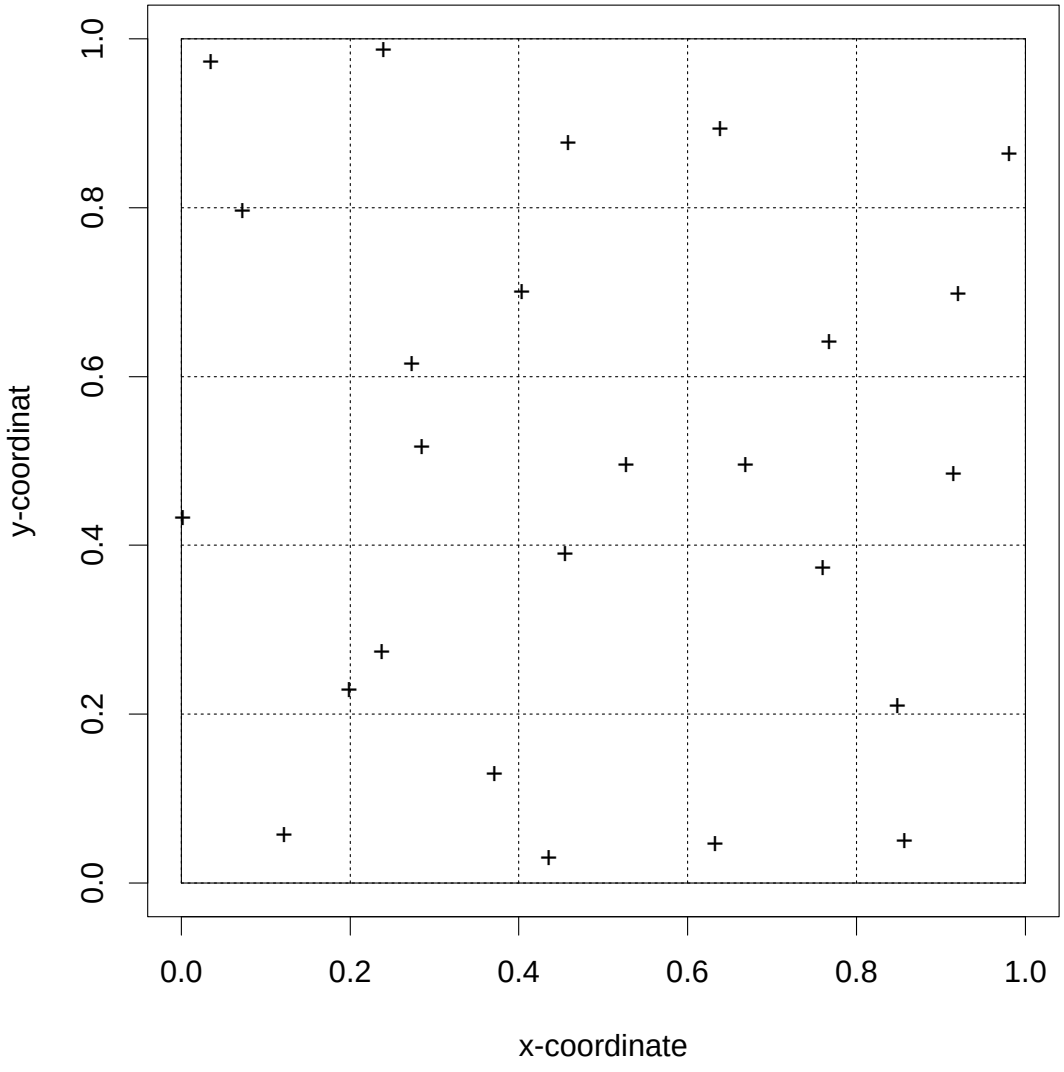
stratified random sampling, n = 25 , blocks = 5



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Stratified random sampling (1)

```
> plot(nrw, col = "lightgrey")
> pts = spsample(nrw, 100, "stratified")
> points(pts, pch = 3)
```

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Stratified random sampling (2)

```
> plot(nrw, col = "lightgrey")
> pts = spsample(nrw, 100, "stratified")
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Stratified random sampling (3)

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Stratified random sampling (4)

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Stratified random sampling (5)

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Stratified random sampling

Approach:

- Put a lattice over the area, with (approximately) n cells
- Randomly sample one point in each cell
- Number of random coordinates: n, constrained to one per lattice cell

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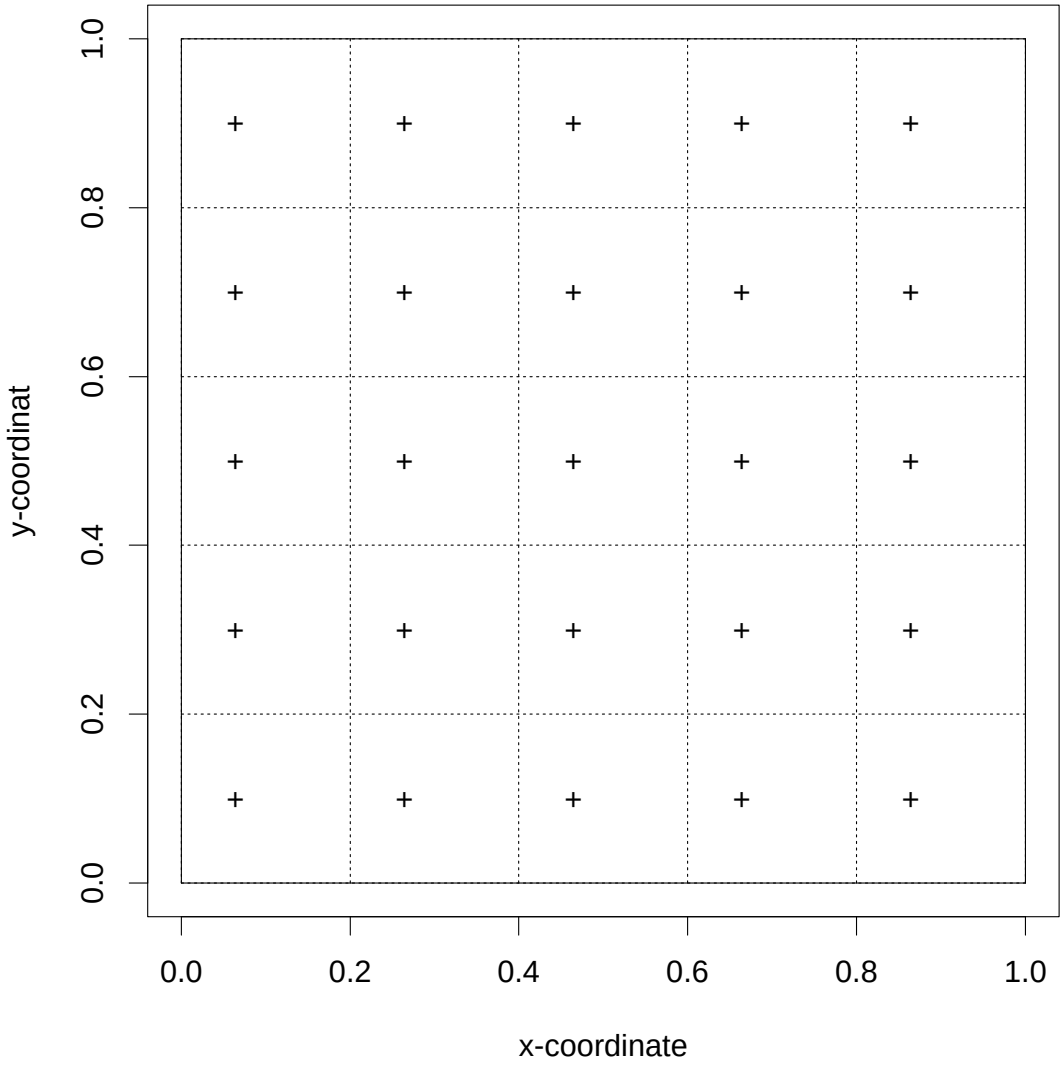
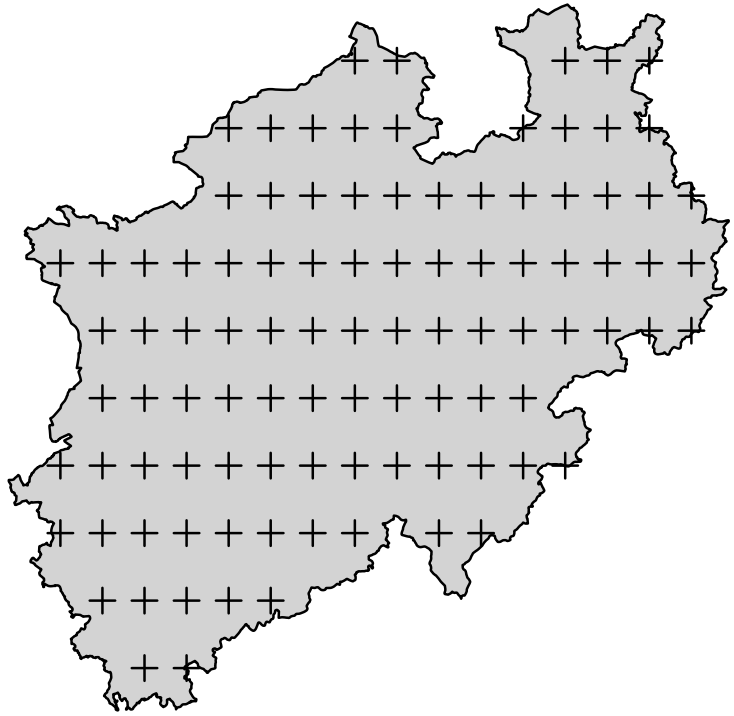
systematic aligned sampling, n = 1 , blocks = 5



Systematically aligned random sampling (1)

```
> plot(nrw, col = "lightgrey")
> pts = spsample(nrw, 100, "regular")
> points(pts, pch = 3)
```



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Systematically aligned random sampling (2)

```
> plot(nrw, col = "lightgrey")
> pts = spsample(nrw, 100, "regular")
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```

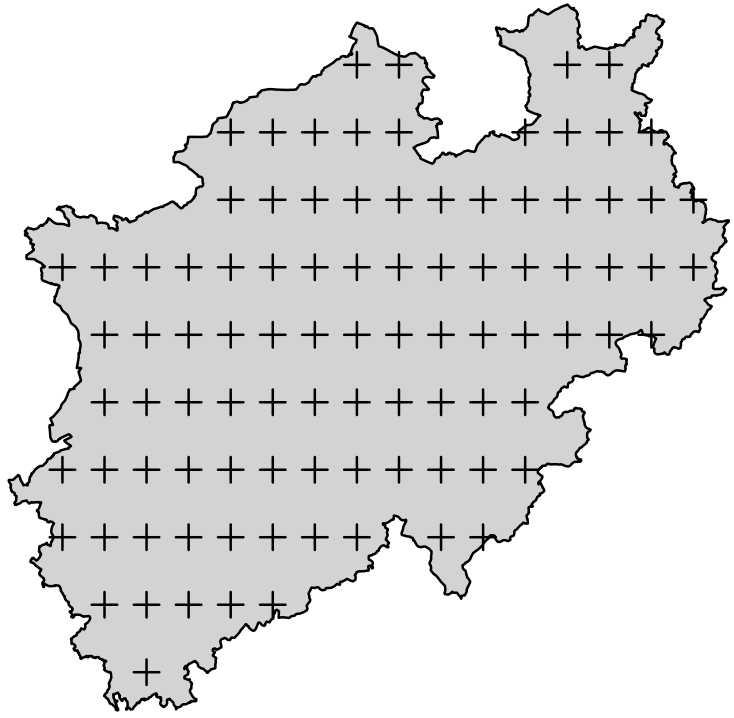

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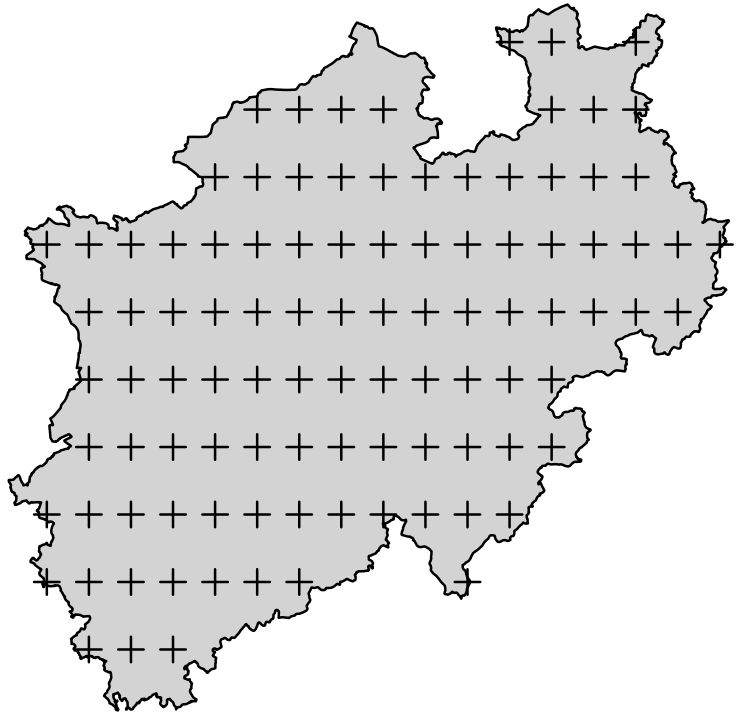

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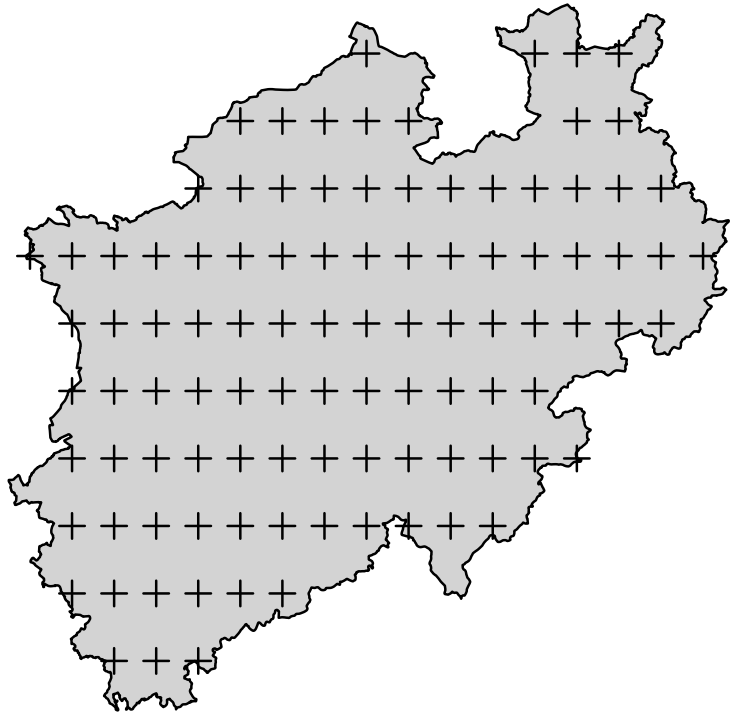

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Systematically aligned (regular) random sampling

Approach:

- Put a lattice over the area, with (approximately) n cells
- Randomly sample one point in the first cell
- Take the same point in all the other cells
- Number of random coordinates: 1

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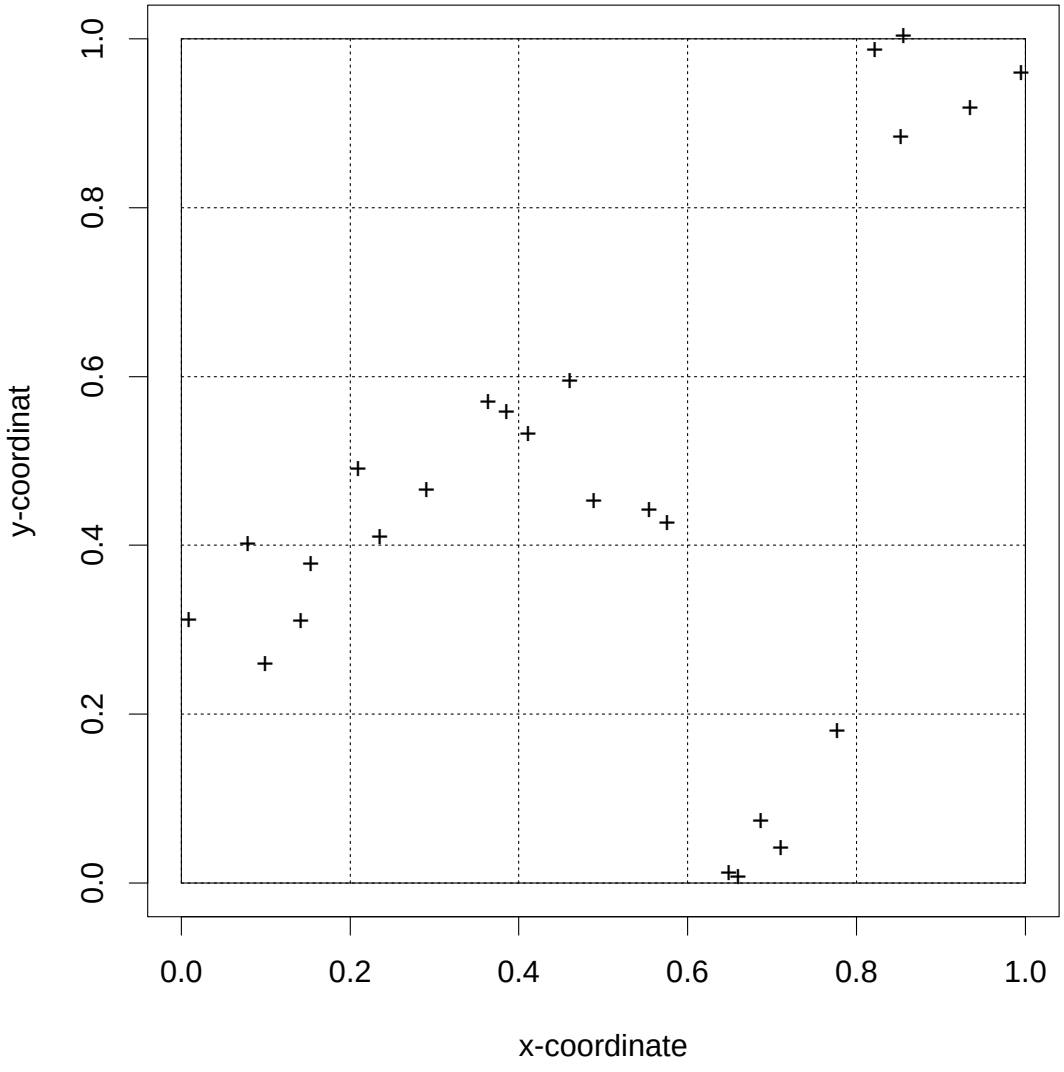
clustered sampling, n = 125 , blocks = 5 , clusters = 5



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Clustered sampling

Possible approach:

- Sample n lattice cells at random,
- Within each lattice cell select m points at random
- results in a sample of size nm
- alternatives: we could apply regular sampling or stratfied sampling within the selected lattice cells
- advantage: travel time
- disadvantage: less efficient: suppose the lattice cells are very small, we may end up with effectively the same information as n single random points.

Alternative sampling methods:

- line sampling, where lines are placed at random (not e.g. existing, such as roads)
- hybrid methods

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Properties of the arithmetic mean

What about the mean value?

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}(X_1 + X_2 + ... + X_n)$$

$$\mathrm{E}\bar{X} = \sum_{i=1}^{n} \mathrm{E}(X_i) = \frac{1}{n}(\mathrm{E}(X_1) + \mathrm{E}(X_2) + ... + \mathrm{E}(X_n))$$

$$= \frac{1}{n}[\mu + \mu + ... + \mu] = \frac{1}{n}[n\mu] = \mu$$

What about it's variability? If all observations are independent, then

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(X_i) = \sigma^2/n$$

with $\sigma^2 = \mathrm{Var}(X)$

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Properties of the arithmetic mean, 2

Remember: standard error of $\bar{X}$, for independent observations is

$$\mathrm{SE} = \frac{\sigma}{\sqrt{n}}$$

- if $n = \infty$, $\mathrm{SE} = 0$ meaning that $\bar{X} = \mu$.
- in other cases, the variability in the difference, $\bar{X} - \mu$ has standard deviation SE
- **ONLY** if $n = 1$, $\mathrm{SE} = \sigma$; this is the case where we take a sample of size 1, and estimate the mean μ by this single value.

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Confidence interval for the mean

If the difference $\bar{X} - \mu$ follows a normal distribution, we can e.g. state that

$$Pr(\bar{X} - 1.96\text{SE} < \mu < \bar{X} + 1.96\text{SE}) = 0.95$$

and form a 95% confidence interval for μ by

$$[\bar{X} - 1.96\text{SE}, \bar{X} + 1.96\text{SE}]$$

Note that in the above $Pr()$ statement, randomness is associated with $\bar{X}$, as this fluctuates from one sample to another, and not with μ.
However, we still don't know σ!

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Point estimation vs interval estimation

- Point estimation is e.g. giving $\bar{X}$ as an estimate of μ
- Obviously, we try always to give the "best" point estimate
- "best" usually has some mathematical connotation: least squares, minimum variance, best linear, maximum likelihood, maximum a-posteriory probability, ...
- A more complete picture is given by the *interval estimate*, where we give the range of likely values for the target parameter (e.g. μ), given sampling error
- this is usually done with a confidence interval that has a certain probability coverage (e.g. 95%)
- probability refers to sampling error/repeated sampling, not to the population parameter (such as μ)

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Confidence intervals, σ known

We saw that

$$Pr(\bar{X} - 1.96\text{SE} < \mu < \bar{X} + 1.96\text{SE}) = 0.95$$

and we can call this a 95% confidence interval.
The essence is that we have limited knowledge about μ, and this is what we can say about it, based on sampling data.
Other probabilities can also be obtained. Let α be the probability that the confidence interval does *not* cover the true value, in this case 0.05.
$z_{\alpha/2}$ is the value of the standard normal curve below which $\alpha/2$ probability lies. Then we obtain a confidence interval with $1 - \alpha$ probability coverage by

$$[\bar{X} + z_{\alpha/2}\text{SE}, \bar{X} + z_{1-\alpha/2}\text{SE}]$$

(Note that $z_{\alpha/2}$ is negative.)
Values for α:

- α should be small, not larger than .1 for the word "confidence" to make sense

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Values for α:

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Confidence intervals, σ known – example

A 99% confidence interval for `Length`, assuming $\sigma = 11$:

```
> load("students.RData")
> attach(students)
> m = mean(Length)
> sd = 11
> se = sd/sqrt(length(Length))
> alpha = 0.01
> c(m + qnorm(alpha/2) * se, m + qnorm(1 - alpha/2) * se)

[1] 175.7123 180.3548

> alpha = 0.05
> c(m + qnorm(alpha/2) * se, m + qnorm(1 - alpha/2) * se)

[1] 176.2673 179.7998

> alpha = 0.1
> c(m + qnorm(alpha/2) * se, m + qnorm(1 - alpha/2) * se)

[1] 176.5513 179.5158
```

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Confidence intervals, σ unknown

What to do if σ is not known (and in real life, it isn't)?
We know that if n is large, we can estimate σ quite well with the sample standard deviation s. If however n is small, the approximation is worse.
We need a distribution that is like the normal distribution, but wider for smaller n. This is what the t-distribution does.

```
> sd = sqrt(var(Length))
> n = length(Length)
> se = sd/sqrt(n)
> alpha = 0.05
> c(m + qnorm(alpha/2) * se, m + qnorm(1 - alpha/2) * se)

[1] 176.2752 179.7919

> c(m + qt(alpha/2, n - 1) * se, m + qt(1 - alpha/2, n -
+     1) * se)

[1] 176.2607 179.8064
```

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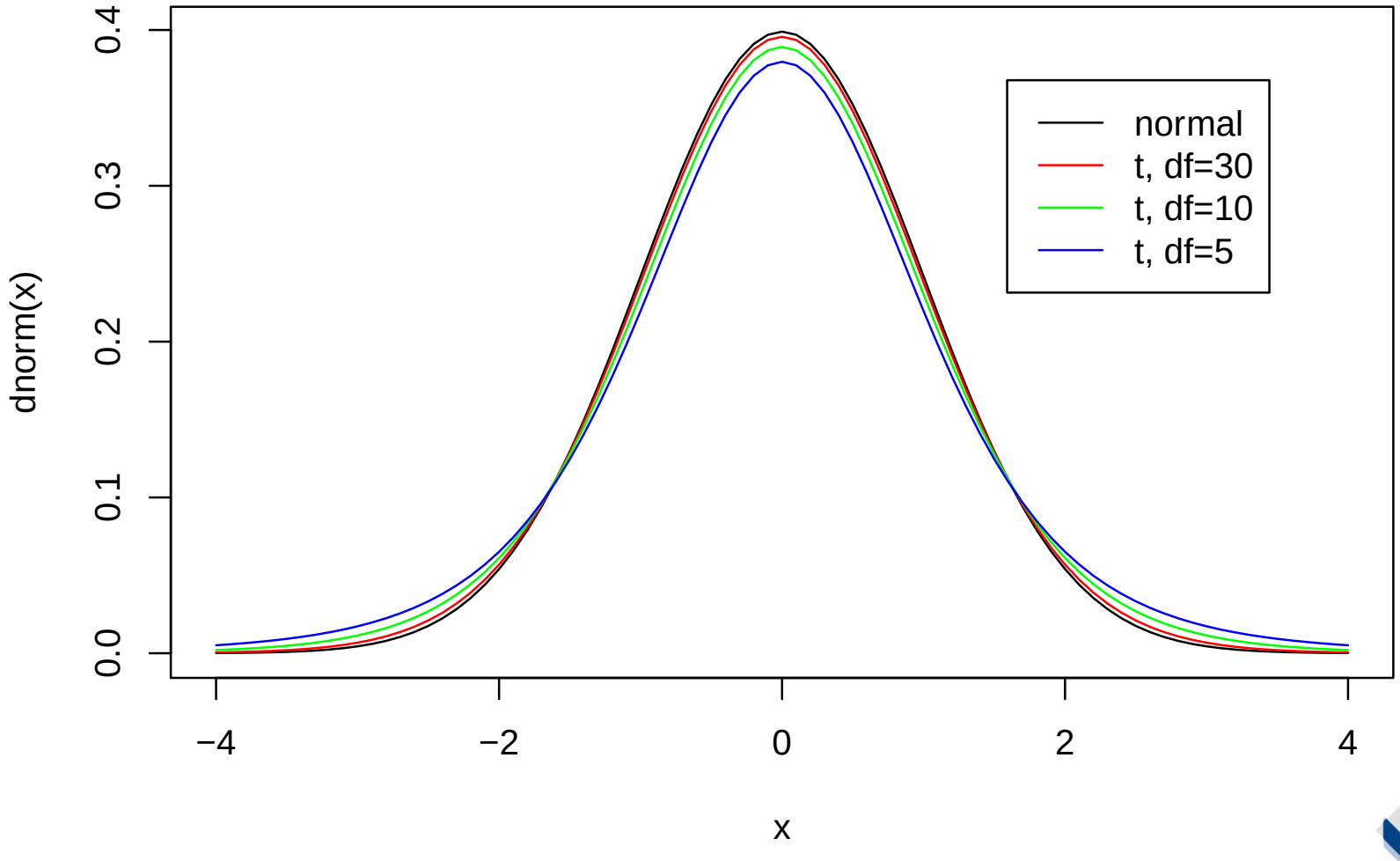
t-distribution



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small sample size:

```
> L10 = Length[1:10]
> m = mean(L10)
> se = sqrt(var(L10)/10)
> c(m + qnorm(alpha/2) * se, m + qnorm(1 - alpha/2) * se)

[1] 159.7252 162.8748

> c(m + qt(alpha/2, 9) * se, m + qt(1 - alpha/2, 9) * se)

[1] 159.4824 163.1176

> L5 = Length[1:5]
> m = mean(L5)
> se = sqrt(var(L5)/5)
> c(m + qnorm(alpha/2) * se, m + qnorm(1 - alpha/2) * se)

[1] 158.4666 159.9334

> c(m + qt(alpha/2, 4) * se, m + qt(1 - alpha/2, 4) * se)

[1] 158.1611 160.2389
```

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